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Twisted Spherical Functions on the Finite Poincaré Upper Half-Plane

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In this note we compute explicit formulae for the twisted spherical functions for the finite analogue of (the double cover of) the classical Poincaré upper half-plane, in any characteristic, and we obtain a uniform description for them resembling the one given by [Curtis (1993, *J. Algebra* **157**, 517–533)] for the characters of the commuting algebra of Gel'fand–Graev representation. We also deduce some character identities.

Key Words: spherical function; harmonic analysis on finite fields.

1. INTRODUCTION

Let F be a finite field with q elements, and let E be its unique quadratic extension. We make no restriction on the characteristic of F.

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Put $G = GL(2, F)$ and denote by K the Coxeter torus of G, realized as the subgroup of all matrices $m_z (z \in E^\times)$ of the maps $m_z\colon w \mapsto zw$ $(w \in E)$ with respect to a fixed F-basis of E. Recall that the finite homogeneous space $\mathscr{H} := E - F \simeq G/K$ may be looked upon as the finite analogue of (the double cover of) the classical Poincaré upper half-plane (see [7]). Classical harmonic analysis on $\mathscr{H}$ means to decompose the induced representation $\mathrm{Ind}_K^G 1$ from the unit character of K to G and to compute the class 1 spherical functions. We are interested here in the "twisted" version of this, i.e., the decomposition of the induced representation $\mathrm{Ind}_K^G \Phi$ from a (necessarily) non-trivial character Φ of K to G. We prove that this representation is multiplicity-free, taking advantage of the fact that this is so for $\mathrm{Ind}_K^G 1$ (Theorem 1). Then, we give an explicit description of the corresponding (twisted) spherical functions (Theorem 2). To this end, we first compute the twisted spherical functions following classical techniques, and, then, thinking of them as characters of the commuting algebra for $\mathrm{Ind}_K^G \Phi$, we rewrite our formulae in a way resembling the character formulae for the commuting algebra of Gel'fand–Graev representation given by Curtis in [2]. The case of characteristic different from 2 has been announced in [10]. Finally, from the fact that certain representations are missing (present) in the decomposition of $L^2(\mathscr{H})$ we deduce a few number theoretical identities, involving the sign characters for $F^\times$ and the group of norm 1 elements in E (Theorem 3).

2. THE REPRESENTATION $\mathrm{Ind}_K^G(\Phi)$ AND ITS COMMUTING ALGEBRA

2.1. *The Induced Representation of* Φ

In this section, unless otherwise explicitly stated, G denotes an arbitrary finite group, K a subgroup of G, and Φ a one-dimensional representation of K. We recall that the spherical functions for the representation $\mathrm{Ind}_K^G(\Phi)$ may be obtained as weighted averages of the characters of G. More precisely,

Definition 1. Let $L^1(G)$ be the group algebra of G, realized as the convolution algebra of all complex functions of G, and let $L^1_\Phi(G, K)$ be the commuting algebra of the induced representation $\mathrm{Ind}_K^G(\Phi)$, realized as the convolution algebra of all complex functions f on G such that

$$f(kgk') = \Phi(k) f(g) \Phi(k')$$

for all $g \in G,\ k, k' \in K$. For $f \in L^1(G)$ put

$$(P_\Phi f)(g) = \frac{1}{|K|} \sum_{k \in K} \Phi^{-1}(k) f(kg)$$

for all $g \in G$.

PROPOSITION 1. (i) *Let χ be the character of an irreducible representation π of G. Then $P_\Phi(\chi)(e) \neq 0$ iff π appears in* $\mathrm{Ind}_K^G \Phi$.

(ii) *The mapping P_Φ is an algebra epimorphism from the center $Z(L^1(G))$ of the convolution algebra $L^1(G)$ onto the center $Z(L^1_\Phi(G, K))$ of the convolution algebra $L^1_\Phi(G, K)$.*

(iii) *In particular, if* $\mathrm{Ind}_K^G(\Phi)$ *is multiplicity free, then P_Φ maps $Z(L^1(G))$ onto the convolution algebra $L^1_\Phi(G, K)$, and P_Φ takes a primitive idempotent to a primitive idempotent or zero.*

(iv) *A double coset KgK is in the support of some primitive idempotent in $L^1_\Phi(G, K)$ if and only if Φ agrees with its conjugate Φ^g on $K \cap g^{-1}Kg$. Here, $\Phi^g(x) = \Phi(gxg^{-1})$.*

Proof. It is easy to check (i), (ii), and (iii). A proof of (iv) can be found in [3, Propositions 11.32 and 11.26). ∎

COROLLARY 1. *The nonzero functions which satisfy the functional equation*

$$h(x)h(y) = \int_K \overline{\Phi(k)} h(xky)dk$$

linearly span the center of the algebra $L^1_\Phi(G, K)$.

Proof. The functions h which satisfy the above functional equation are exactly the complex multiples of the functions $P_\Phi(\chi)$; for a proof (see [12]). Therefore, the corollary follows. ∎

THEOREM 1. *For $G = GL(2, F)$ and $K = \{m_z, z \in E^\times\}$ the Coxeter torus, the commuting algebra $L^1_\Phi(G, K)$ is commutative, and its dimension is q whenever $\Phi = \Phi^q$ and $q - 1$ otherwise, where $\Phi^q(m_z) := \Phi(m_{z^q})$ for all $z \in E^\times$.*

We will prove this theorem in Section 5.

3. THE IRREDUCIBLE REPRESENTATIONS OF $G = GL(2, F)$

We follow [5] or [8]. For Θ, a character of $F^\times \times F^\times$ or a character of $E^\times$, we denote by π^d_Θ the irreducible representation of $G = GL(2, F)$ of dimension d and character parameter Θ. We denote by $z \mapsto \bar{z} = z^q$ the Frobenius automorphism of E over F, and for any character Λ of $E^\times$ we put $\Lambda^q(z) = \Lambda(z^q)$. Let N denote the norm of E over F. Then we have

(i) $\pi^d_\Theta = \pi^{q+1}_{\alpha,\beta} \simeq \pi^{q+1}_{\beta,\alpha}$ (principal series with parameter $(\alpha, \beta) \in (F^\times \times F^\times)^\wedge$, $\alpha \neq \beta$);

(ii) $\pi^d_\Theta = \pi^q_\alpha$ (Steinberg representation with parameter $\alpha \in (F^\times)^\wedge$);

(iii) $\pi_{\Theta}^{d} = \pi_{\alpha}^{1} = \alpha \circ det$ (one-dimensional representation with parameter $\alpha \in (F^{\times})^{\wedge}$);

(iv) $\pi_{\Theta}^{d} = \pi_{\Lambda}^{q-1} \simeq \pi_{\Lambda^q}^{q-1}$ (discrete series with parameter $\Lambda \in (E^{\times})^{\wedge}$, $\Lambda \neq \Lambda^q$).

We extend our notations for the degenerate cases, i.e., for $\alpha = \beta$ and $\Lambda = \Lambda^q$, by putting

$$\pi_{\alpha,\,\alpha}^{q+1} = \pi_{\alpha}^{q} + \pi_{\alpha}^{1}, \quad \pi_{\alpha\circ N}^{q-1} = \pi_{\alpha}^{q} - \pi_{\alpha}^{1} \qquad (\alpha \in (F^{\times})^{\wedge}).$$

We set in every case χ_{Θ}^{d} for the character of the representation π_{Θ}^{d}. Thus, we have (cf. [5])

$$\chi_{\alpha,\,\alpha}^{q+1} = \chi_{\alpha}^{q} + \chi_{\alpha}^{1}, \quad \chi_{\alpha\circ N}^{q-1} = \chi_{\alpha}^{q} - \chi_{\alpha}^{1} \qquad (\alpha \in (F^{\times})^{\wedge}).$$

4. THE FINITE POINCARÉ UPPER HALF-PLANE $\mathscr{H}$.

From now on, we will only consider the transitive action of $G = GL(2, F)$ on $\mathscr{H}$ by homographic transformations. That is, we have $z \mapsto g \cdot z = \frac{az+b}{cz+d}$ for $z \in \mathscr{H}$ and $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2, F)$. The action leaves invariant the $F \cup \{\infty\}$-valued "distance" D given by

$$D(z, w) = \frac{N(z - w)}{N(z - \bar{w})} \qquad \text{for } z, w \in \mathscr{H}$$

with the convention that $D(z, \bar{z}) = \infty$. Then, we have (see 7, 11).

PROPOSITION 2. Image$(D) = \{\infty\} \cup F - \{1\}$, *and D is an orbit classifying symmetric invariant for the homographic action of G in $\mathscr{H} \times \mathscr{H}$.* ∎

A consequence of this proposition is that the intertwining algebra $\mathrm{End}_G(L^2(\mathscr{H})$ of the natural representation of $GL(2, F)$ in $L^2(\mathscr{H})$ is commutative, and so $L^2(\mathscr{H})$ is multiplicity-free; in other words, $(GL(2, F), K)$ is a Gel'fand pair. We recall since that $\mathscr{H} \simeq G/K$, we have

$$L^2(\mathscr{H}) \simeq \mathrm{Ind}_K^G \mathbf{1} \quad \text{and} \quad \mathrm{End}_G(L^2(\mathscr{H})) \simeq L^1_{\mathbf{1}}(G, K).$$

5. THE INTERTWINING ALGEBRA IN THE TWISTED CASE

5.1. *The Multiplicity One Property*

Let ϕ be the restriction of Φ to $F^{\times}$. We prove that if the restriction to K of an irreducible representation of G is twisted by the generalized character $\Phi + \Phi^q$ of K, one obtains the restriction to K of the sum of two irreducible representations of G. More precisely:

LEMMA 1. *On K, we have*

$$(\Phi+\Phi^q)\chi_\alpha^q = \chi_{\Phi(\alpha\circ N)}^{q-1} + \chi_{\phi\alpha,\,\alpha}^{q+1} \qquad (\Phi+\Phi^q)\chi_\alpha^1 = \chi_{\phi\alpha,\,\alpha}^{q+1} - \chi_{\Phi(\alpha\circ N)}^{q-1}$$

$$(\Phi+\Phi^q)\chi_{\alpha,\,\beta}^{q+1} = \chi_{\phi\alpha,\,\beta}^{q+1} + \chi_{\alpha,\,\phi\beta}^{q+1} \qquad (\Phi+\Phi^q)\chi_\Lambda^{q-1} = \chi_{\Phi\Lambda}^{q-1} + \chi_{\Phi^q\Lambda}^{q-1}$$

The lemma readily follows from the character table in [5]. ∎

PROPOSITION 3. *The representation* $\mathrm{Ind}_K^G\Phi$ *is multiplicity-free.*

Proof. Indeed, since the restriction to K of any character χ of G is Frobenius invariant, it follows that the multiplicity $m_\Phi(\pi)$ of π in $\mathrm{Ind}_K^G\Phi$ equals $\frac{1}{2}|K|^{-1}\sum_K(\bar{\Phi}+\bar{\Phi}^q)\chi_\pi$, that is, by Lemma 1, the average of the multiplicities in $\mathrm{Ind}_K^G\mathbf{1}$ of two representations of G (one of which may be virtual!). Notice then that the multiplicities of the irreducible representations of G in $\mathrm{Ind}_K^G\mathbf{1}$ are at most 1 and that the multiplicities for the degenerated cases are given by

$$m_1(\pi_{\alpha,\,\alpha}^{q+1}) = \delta_{\alpha^2,\,1}, \quad m_1(\pi_{\alpha\circ N}^{q-1}) = \delta_{\alpha,\,\epsilon} - \delta_{\alpha,\,1} \qquad (\alpha\in(F^\times)^\wedge).$$

∎

In Table I, λ denotes the restriction of the character Λ to $(F^\times)^\wedge$.

5.2. *Proof of Theorem* 1.

We have already shown that the intertwining algebra is abelian. Although its dimension may be computed from Table I, we prefer to give here a direct proof.

Let S be any set of representatives for $K\backslash G/K$. Then the Intertwining Number Theorem for induced representations says that $\dim L_\Phi^1(G,K)$ is equal to the number of $s\in S$ such that $\Phi=\Phi^s$ on $K\cap s^{-1}Ks$. Here, $\Phi^s(k) = \Phi(sks^{-1})$. We recall that in our case we have $|K\backslash G/K| = |\mathrm{Im}(D)| = q$, and $|N_G(K)/K| = 2$. To construct such a set S, let A_1 denote the group of all matrices of the form $\left(\begin{smallmatrix} x & y \\ 0 & 1\end{smallmatrix}\right)$ for $x\in F^\times$, $y\in F$. Then we may choose our set S as a subset of A_1. In that case S will contain the identity and a unique representative s_0 of the non-trivial coset in $N_G(K)/K$.

A straightforward computation, for each $s\in S$, leads us to

(i) $K\cap s^{-1}Ks = F^\times$ for $s\neq 1, s_0$.

TABLE 1
Multiplicity $m_\Phi(\pi_\Theta^d)$ of π_Θ^d in $\mathrm{Ind}_K^G\Phi$ ($\Phi\in(E^\times)^\wedge$)

$m_\Phi(\pi_{\alpha,\,\beta}^{q+1})$	$m_\Phi(\pi_\alpha^q)$	$m_\Phi(\pi_\alpha^1)$	$m_\Phi(\pi_\Lambda^{q-1})$
$\delta_{\alpha\beta,\,\phi}$	$\delta_{\alpha^2,\,\phi} - \delta_{\alpha\circ N,\,\Phi}$	$\delta_{\alpha\circ N,\,\Phi}$	$\delta_{\lambda,\,\phi} - \delta_{\Lambda,\Phi} - \delta_{\Lambda^q,\,\Phi}$

(ii) $K \cap s^{-1}Ks = K$ for $s = 1, s_0$.

Whence, for $s \in S$ and $s \neq s_0$, we have that $\Phi = \Phi^s$ on $K \cap s^{-1}Ks$. Thus, $\dim L^1_\Phi(G, K) \geq q - 1$. Now, $m_{\bar{z}}$ is conjugated to m_z by s_0, and thus $\Phi^{s_0} = \Phi^q$. Hence, the representative s_0 contributes to $\dim L^1_\Phi(G, K)$ if and only if $\Phi = \Phi^q$. ∎

6. EXPLICIT FORMULAE FOR THE TWISTED SPHERICAL FUNCTIONS

Let us denote by $F^\sharp$ the subset of F consisting of all non-zero squares (resp. all elements of the form $a^2 + a$ $(a \in F)$) if char $F \neq 2$ (resp. if char $F = 2$). We fix a non-zero $t_0 \in F^\times - F^\sharp$ and $\theta \in E$, a root of the equation

$$x^2 = t_0 \quad \text{if char } F \neq 2 \qquad (\text{resp. } x^2 + x = t_0 \quad \text{if char } F = 2).$$

Notice that then

$$\bar{\theta} = -\theta \quad (\text{if char } F \neq 2), \qquad \bar{\theta} = \theta + 1 \quad (\text{if char } F = 2).$$

From now on, the matrix of an F-linear operator of the space E is computed with respect to the F-basis $\{1, \theta\}$. The isotropy group of θ, in any characteristic, turns out to be K.

We note that every $g \in G := GL(2, F)$ may be written in a unique way as $g = kg_x$ with $k \in K$, $x \in \mathscr{H}$, where

$$g_x = \begin{pmatrix} x_2 & x_1 \\ 0 & 1 \end{pmatrix}, \qquad \text{with } x = x_1 + \theta x_2 = g_x \cdot \theta \quad (x_1 \in F, x_2 \in F^\times).$$

Moreover, we set $\rho = D(x, \theta)$ and $s = \frac{1+\rho}{1-\rho}$.

Since $K = \{m_z, z \in E^\times\} \simeq E^\times$, the dual group to K is isomorphic to the dual group of $E^\times$. Thus, a character $\Phi \in (E^\times)^\wedge$ gives rise to a character of K. To avoid a cumbersome notation we write $\Phi(z)$ instead of $\Phi(m_z)$. We fix $\Phi \in K^\wedge$ and let χ be the character of an irreducible representation of G occurring in $\mathrm{Ind}_K^G \Phi$. Then Proposition 1(iii) implies that the spherical function φ_χ^Φ corresponding to χ is, for $g = kg_x \in G (k \in K, x \in \mathscr{H})$, given by

$$\varphi_\chi^\Phi(g) = \frac{\Phi(k)}{q^2 - 1} \sum_{z \in E^\times} \overline{\Phi(z)} \chi(m_z g_x). \tag{1}$$

The fact that $\mathrm{Ind}_K^G(\Phi)$ is a multiplicity-free representation implies that $\varphi_\chi^\Phi(Id) = 1$.

Let A denote a semisimple associative algebra of dimension 2 over the field F. Therefore, $A \cong F \times F$ or $A \cong E$. We set $\mathrm{sgn}_A = 1$ (resp. -1)

for $A = F \times F$ (resp. $A = E$). Let $a \mapsto \bar{a}$ denote the conjugation of A with respect to F. Then, the conjugation is given by $\bar{a} = a^q$ if $A = E$, and $\bar{a} = (a_2, a_1)$ for $a = (a_1, a_2) \in F \times F$. In both cases, we identify the field F with the subfield of fixed points for the conjugation of A with respect to F.

As usual, for $a \in A$, we denote its trace (resp. norm) by $\mathrm{tr}_A(a) = a + \bar{a}$, (resp. $n_A = a\bar{a}$), and we put $\mathcal{U}(a) = a\bar{a}^{-1}$. Moreover, we set $\mathrm{tr}_E = \mathrm{Tr}$, $n_E = N$.

For $\rho \neq \infty$ and $a \in A^\times$, we define $\mathrm{Sol}^A(\rho, a)$ to be the set of solutions $z \in E$ to the system

$$\mathrm{Tr}(z) = \mathrm{tr}_A(a), \quad N(z) = n_A(a)(1-\rho)^{-1}.$$

Notice that $\mathrm{Sol}^A(\rho, \bar{a}) = \mathrm{Sol}^A(\rho, a)$, hence $\mathrm{Sol}^A(\rho, a)$ depends only on $\mathcal{U}(a)$, and that

$$|\mathrm{Sol}^A(\rho, a)| \leq 2.$$

More precisely, if char $F \neq 2$ we show in Section 7 that

$$|\mathrm{Sol}^A(\rho, a)| = 1 - \epsilon\left[n_A(a)\left(\mathrm{tr}_A\left(\mathcal{U}(a) - \frac{1+\rho}{1-\rho}\right)\right)\right] \qquad (a \notin F), \quad (2)$$

$$|\mathrm{Sol}^A(\rho, a)| = 1 - \epsilon\left(\frac{\rho}{\rho - 1}\right) \qquad (a \in F^\times). \quad (3)$$

Here, ϵ is the sign character for F, which equals 1 on $F^\sharp$, vanishes at 0, and equals -1 on $F^\times - F^\sharp$. If char $F = 2$, we show in Section 7 that

$$|\mathrm{Sol}^A(\rho, a)| = 1 - \epsilon[(\mathrm{tr}_A((1-\rho)\mathcal{U}(a)))^{-1}] \qquad (a \notin F), \quad (4)$$

and

$$\mathrm{Sol}^A(\rho, a) = \left\{\sqrt{\frac{n_A(a)}{(1-\rho)}}\right\} = \left\{\frac{a}{\sqrt{(1-\rho)}}\right\} \qquad (a \in F^\times), \quad (5)$$

where $\sqrt{t}$ denotes the unique square root of any $t \in F$, and ϵ is the sign character for F^+, which equals 1 on $F^\sharp$, and -1 on $F^+ - F^\sharp$.

In particular, we have that $\mathrm{Sol}^E(0, a) = \{a, \bar{a}\}$, for any $a \in E$ and that $\mathrm{Sol}^{F \times F}(0, a) = \{a\}$ if $a \in F^\times$ and is empty otherwise.

Let $\mathrm{Sol}_0^A(\infty, a)$ stands for the set of all $z \in E$ such that $N(z) = -n_A(a)$, for $a \in A$, with $\mathrm{tr}_A(a) = 0$. For further use, we write $\delta_{x,y} = 1$ if $x = y$ and $\delta_{x,y} = 0$ otherwise.

Next, we consider the special functions S_Φ^A defined by

$$S_\Phi^A(\rho, a) = \sum_{z \in \mathrm{Sol}^A(\rho, a)} \overline{\Phi}(z) \qquad (\rho \in \mathrm{Im}(D), \rho \neq 0, \infty, a \in A), \quad (6)$$

$$S_\Phi^A(0, a) = (\mathrm{sgn}(A)q + \mathbf{1})\overline{\Phi}(a) \qquad (a \in F), \quad (7)$$

$$S_{\Phi}^{A}(0,a)=\sum_{z\in \mathrm{Sol}^{A}(0,a)}\overline{\Phi}(z)\qquad (a\in A,\, a\neq \bar{a}),\tag{8}$$

$$S_{\Phi}^{A}(\infty,a)=\delta_{\mathrm{tr}_A(a),\,0}\sum_{z\in \mathrm{Sol}_0^{A}(\infty,a)}\overline{\Phi}(z).\tag{9}$$

To describe the spherical functions φ_χ^Φ occurring in $\mathrm{Ind}_K^G(\Phi)$, i. e., such that χ is the character of a representation π of G appearing in $\mathrm{Ind}_K^G(\Phi)$, we set $\Phi(0)=1$, and we recall that $\Phi=\Phi^q$ if and only if there exists $\phi_0\in(F^\times)^\wedge$ so that $\Phi=\phi_0\circ N$.

THEOREM 2. *For each character ξ of $A^\times$ and for each $g=kg_x\in G$ $(k\in K, x\in\mathscr{H})$, we let $x'=\frac{x+\theta}{2\theta}$ if* char $F\neq 2$ *(resp. $x'=\bar{x}+\theta$ if* char $F=2$), *and we set*

$$\varphi_\xi^A(g)=\frac{\mathrm{sgn}(A)}{q^2-1}\Phi(kx')\Bigg[\sum_{a\in A^\times}S_\Phi^A(\rho,a)\xi(a)\Bigg].$$

Then, the spherical functions φ_χ^Φ are given by the following formulae:

(i) *Discrete series*: $\varphi^\Phi_{\chi_\wedge^{q-1}}=\varphi_\xi^A$ *for $A=E$ and $\xi=\Lambda\in(E^\times)^\wedge$, $\Lambda\neq\Lambda^q$. In particular, when* char $F\neq 2$, *we have that* $\varphi^\Phi_{\chi_\wedge^{q-1}}(g_{\bar\theta})=-\frac{\Lambda}{\Phi}(\theta)\phi_0(-1)$ $\delta_{\Phi,\,\Phi^q}$, *whereas for* char $F=2$, $\varphi^\Phi_{\chi_\wedge^{q-1}}(g_{\bar\theta})=-1$.

(ii) *Principal series*: $\varphi^\Phi_{\chi_{\alpha,\beta}^{q+1}}=\varphi_\xi^A$ *for $A=F\times F$ and $\xi=(\alpha,\beta)\in(F^\times\times F^\times)^\wedge$, with $\alpha\neq\beta$. In particular, for* char $F\neq 2$, *we have* $\varphi^\Phi_{\chi_{\alpha,\beta}^{q+1}}(g_{\bar\theta})=\alpha(-1)\delta_{\Phi,\,\Phi^q}$, *and* $\varphi^\Phi_{\chi_{\alpha,\beta}^{q+1}}(g_{\bar\theta})=1$, *for* char $F=2$.

(iii) *Steinberg and one-dimensional representations*: *For a two-dimensional associative semisimple algebra A over the field F and for a character α of $F^\times$, let $\alpha_A=\alpha\circ n_A$. In the formula*

$$\varphi^\Phi_{\chi_\alpha^q}(g)=\tfrac{1}{2}\Bigg[\sum_A\varphi^A_{\alpha_A}\Bigg],\qquad \varphi^\Phi_{\chi_\alpha^1}(g)=\tfrac{1}{2}\Bigg[\sum_A\mathrm{sgn}(A)\varphi^A_{\alpha_A}\Bigg],$$

$\sum_A$ runs over a set of representatives of the equivalence classes of the two-dimensional associative, semisimple algebras over F; that is,

$$\varphi^\Phi_{\chi_\alpha^q}(g)=\frac{1}{2(q^2-1)}\Phi(kx')\Bigg[\sum_A\mathrm{sgn}(A)\sum_{a\in A^\times}S_\Phi^A(\rho,a)(\alpha\circ n_A)(a)\Bigg],$$

$$\varphi^\Phi_{\chi_\alpha^1}(g)=\frac{1}{2(q^2-1)}\Phi(kx')\Bigg[\sum_A\sum_{a\in A^\times}S_\Phi^A(\rho,a)(\alpha\circ n_A)(a)\Bigg].$$

Furthermore,

$$\varphi^\Phi_{\chi_\alpha^1}(g)=\Phi(kx')\alpha(1-\rho)\,\delta_{\Phi,\,\alpha\circ N}.$$

For char $F \neq 2$

$$\varphi^{\Phi}_{\chi^q_\alpha}(g_{\bar\theta}) = \tfrac{1}{2}\left[1 - \frac{\alpha}{\phi_0}(t_0)\right]\alpha(-1)\delta_{\Phi,\,\Phi^q},$$

$$\varphi^{\Phi}_{\chi^1_\alpha}(g_{\bar\theta}) = \tfrac{1}{2}\left[1 + \frac{\alpha}{\phi_0}(t_0)\right]\alpha(-1)\delta_{\Phi,\,\Phi^q},$$

and when char $F = 2$,

$$\varphi^{\Phi}_{\chi^q_\alpha}(g_{\bar\theta}) = 0, \qquad \varphi^{\Phi}_{\chi^1_\alpha}(g_{\bar\theta}) = 1.$$

We will prove this theorem in Section 7.

Remark 1. We now look upon the above theorem in the way Curtis [2] described the characters of the commuting algebra of Gel'fand–Graev representation. So, we regard the spherical functions φ^{Φ}_{χ} as characters of the algebra $L^1_{\Phi}(G, K)$. For each of our algebras A, we define a map $\mathscr{C}_A$ from G to $L^1(A^\times)$ by the rule

$$\mathscr{C}_A(g)\colon a \mapsto \frac{\operatorname{sgn}(A)}{q^2-1}\Phi(kx')S^A_{\Phi}(\rho, a),$$

for $g = kg_x \in G$. Then the character of the algebra $L^1_{\Phi}(G, K)$ associated to the spherical function φ_ξ may be written as the composition of the algebra homomorphism

$$\tilde{\mathscr{C}}_A\colon f \mapsto |G|^{-1}\sum_{g\in G} f(g)\mathscr{C}_A(g)$$

from $L^1_{\Phi}(G, K)$ to $L^1(A^\times)$ with the canonical extension of ξ to an algebra homomorphism from $L^1(A^\times)$ to $\mathbb{C}$.

7. PROOF OF THE FORMULAE IN THEOREM 2, ANY CHARACTERISTIC

In this section, we write down and prove the proposed formulae for the twisted spherical functions φ^{Φ}_{χ} in any characteristic. From now on, we drop the upper index Φ and write φ_χ instead of φ^{Φ}_{χ}.

Let $x = x_1 + \theta x_2$, ρ be as in Section 6. Let Φ be a character of $K = \{m_z\colon z \in E^\times\} \simeq E^\times$ and let χ be the character of an irreducible representation of G appearing in $\operatorname{Ind}^G_K(\Phi)$, so that the corresponding spherical function φ_χ of type Φ associated to χ is non-zero. Since

$$\varphi_\chi(g_x) = \frac{1}{q^2-1}\sum_{z\in E^\times}\overline{\Phi(z)}\chi(m_z g_x), \tag{1}$$

we have for $x = \theta$ that $\varphi_\chi(g_\theta) = \dim \mathrm{Hom}_K((\mathbb{C}, \Phi), (V, \chi)) = 1$. We notice that $\chi(m_a g) = \Phi(a)\chi(g)$ for $a \in F^\times$, $g \in G$.

To begin with, we compute $\varphi_\chi(g_{\bar\theta})$. To this end, we fix $z_0 \in E$ so that $N(z_0) = t_0$. We denote by det (resp. tra) the matrix determinant (resp. trace).

LEMMA 2. *We have*

$$\varphi_\chi(g_{\bar\theta}) = \tfrac{1}{2}\delta_{\Phi,\,\Phi^q}\big[\chi(g_{\bar\theta}) + \overline{\Phi(z_0)}\chi(m_\theta)\big]$$

if char $F \neq 2$, *and*

$$\varphi_\chi(g_{\bar\theta}) = \delta_{\Phi,\Phi^q}[\chi(g_{\bar\theta})]$$

if char $F = 2$.

Proof. Let us recall that for all $z = z_1 + z_2\theta \in E$ we have

$$m_z = \begin{pmatrix} z_1 & t_0 z_2 \\ z_2 & z_1 \end{pmatrix} \qquad \left(\text{resp. } m_z = \begin{pmatrix} z_1 & t_0 z_2 \\ z_2 & z_1 + z_2 \end{pmatrix}\right)$$
$$g_{\bar\theta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \left(\text{resp. } g_{\bar\theta} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)$$

if char $F \neq 2$ (resp. char $F = 2$), and so,

$$m_z g_{\bar\theta} = \begin{pmatrix} -z_1 & t_0 z_2 \\ -z_2 & z_1 \end{pmatrix} \qquad \left(\text{resp. } m_z g_{\bar\theta} = \begin{pmatrix} z_1 & z_1 + t_0 z_2 \\ z_2 & z_1 \end{pmatrix}\right)$$

if char $F \neq 2$ (resp. if char $F = 2$).

Thus, in any case, $\mathrm{tra}(m_z g_{\bar\theta}) = 0$, and $\det(m_z g_{\bar\theta}) = -N(z)$. Now, two non-scalar 2×2 matrices are conjugated if and only if they have the same trace and determinant. So, when char $F \neq 2$, we have that, for $a \in F^\times$, $u \in U := \mathrm{Ker}(N)$,

(i) $m_{au} g_{\bar\theta}$ is conjugated to $a g_{\bar\theta}$,

whereas

(ii) $m_{z_0 au} g_{\bar\theta}$ is conjugated to $m_{a\theta}$.

Notice that $E^\times = F^\times U \cup z_0 F^\times U$. Now, $m_{au} = m_{bv}$ (resp. $m_{z_0 au} = m_{z_0 bv}$), $a, b \in F^\times$, $u, v \in U$ if and only if $(a, u) = \pm(b, v)$. Therefore, for any function h from $E^\times$ into $\mathbb{C}$, we have that

$$\sum_{z \in E^\times} h(z) = \tfrac{1}{2}\Bigg[\sum_{a \in F^\times,\, u \in U} h(au) + \sum_{a \in F^\times,\, u \in U} h(z_0 au)\Bigg].$$

Thus,

$$\begin{aligned}
\varphi_\chi(g_{\bar\theta}) &= \frac{1}{2(q^2-1)} \sum_{a\in F^\times,\, u\in U} \overline{\Phi(au)}\Phi(a)\chi(g_{\bar\theta}) + \overline{\Phi(z_0)\Phi(au)}\Phi(a)\chi(m_\theta) \\
&= \frac{1}{2(q^2-1)} \Bigg(\sum_{u\in U} \overline{\Phi(u)}\Bigg)\Bigg(\sum_{a\in F^\times} \overline{\Phi(a)}\Phi(a)\Bigg)\Bigg(\chi(g_{\bar\theta}) + \overline{\Phi(z_0)}\chi(m_\theta)\Bigg) \\
&= \tfrac{1}{2}\delta_{\Phi,\,\Phi^q}\Bigg(\chi(g_{\bar\theta}) + \overline{\Phi(z_0)}\chi(m_\theta)\Bigg).
\end{aligned}$$

On the other hand, if we assume that char $F = 2$, then the non-scalar matrix $m_z g_{\bar\theta}$ is always conjugated to $[\sqrt{N(z)}]g_{\bar\theta}$, so that

$$\begin{aligned}
\varphi_\chi(g_{\bar\theta}) &= \frac{1}{(q^2-1)} \sum_{z\in E^\times} \overline{\Phi(z)}\chi\Big(\Big[\sqrt{N(z)}\Big]g_{\bar\theta}\Big) \\
&= \frac{1}{(q^2-1)} \sum_{z\in E^\times} \overline{\Phi(z)}\chi\Big(\Big[\sqrt{N(z)}\Big]g_{\bar\theta}\Big) \\
&= \frac{1}{(q^2-1)} \sum_{z\in E^\times} \left[\frac{\Phi(\sqrt{N(z)})}{\Phi(z)}\right]\chi(g_{\bar\theta}) = \delta_{\Phi,\,\Phi^q}\chi(g_{\bar\theta}). \quad \blacksquare
\end{aligned}$$

Notice that Hilbert's Satz 90 implies that Φ restricted to U is the trivial character if and only if $\Phi = \Phi^q$, i.e., if and only if $\Phi = \phi_0 \circ N$, where ϕ_0 is a character of $F^\times$.

We now compute $\varphi_\chi(g_{\bar\theta})$ for each character χ. We assume that $\Phi = \Phi^q$ and write $\Phi = \phi_0 \circ N$. For this, we make use of the table in [5, p. 64].

For a discrete series representation we have, for char $F \neq 2$, that

$$\varphi_{\chi_\Lambda^{q-1}}(g_{\bar\theta}) = \tfrac{1}{2}\overline{\phi_0(t_0)}[-(\Lambda+\Lambda^q)](\theta).$$

But since $\theta^q = -\theta$ and Φ restricted to F is equal to Λ restricted to F, we obtain $\Lambda(-1) = \Phi(-1) = \phi_0(N(-1)) = 1, \Phi(z_0) = \phi_0(t_0), \Phi(\theta) = \phi_0(-t_0)$. Thus, we have

$$\varphi_{\chi_\Lambda^{q-1}}(g_{\bar\theta}) = -\frac{\Lambda(\theta)}{\phi_0(t_0)} = -\frac{\Lambda}{\Phi}(\theta)\phi_0(-1) \in \{1,-1\}.$$

When char $F = 2$, we obtain $\varphi_{\chi_\Lambda^{q-1}}(g_{\bar\theta}) = \chi_\Lambda^{q-1}(g_{\bar\theta}) = -1$.

For a principal series representation, we have, when char $F \neq 2$, that

$$\varphi_{\chi_{\alpha,\beta}^{q+1}}(g_{\bar\theta}) = \tfrac{1}{2}[\alpha(1)\beta(-1) + \alpha(-1)\beta(1)] = \alpha(-1),$$

since $\alpha(-1)\beta(-1) = \Phi(-1) = \phi_0(N(-1)) = 1$.

In the case where char $F = 2$, we get $\varphi_{\chi_{\alpha,\beta}^{q+1}}(g_{\bar\theta}) = \chi_{\alpha,\beta}^{q+1}(g_{\bar\theta}) = 1$.

For the Steinberg representations, we have, when char F $\neq 2$,

$$\begin{aligned}\varphi_{\chi_\alpha^q}(g_{\bar\theta}) &= \frac{1}{2}\big[\alpha(-1)+\overline{\Phi(z_0)}(-1)\alpha(N(\theta))\big]\\ &= \frac{1}{2}\big[\alpha(-1)-\overline{\phi_0(t_0)}\alpha(-t_0)\big] = \frac{1}{2}\alpha(-1)\left[1-\frac{\alpha(t_0)}{\phi_0(t_0)}\right],\end{aligned}$$

since $N(z_0)=t_0=-N(\theta)$.

On the other hand, when char $F=2$, we get $\varphi_{\chi_\alpha^q}(g_{\bar\theta})=0$.

For the one-dimensional representations, we have

$$\varphi_{\chi_\alpha^1}(g_{\bar\theta}) = \frac{1}{2}\big[\alpha(-1)+\overline{\Phi(z_0)}\alpha(N(m_\theta))\big] = \frac{1}{2}\alpha(-1)\left[1+\frac{\alpha(t_0)}{\phi_0(t_0)}\right],$$

when char $F\neq 2$, and $\varphi_{\chi_\alpha^1}(g_{\bar\theta})=\chi_\alpha^1(g_{\bar\theta})=1$, for char $F=2$. Thus, we have computed $\varphi_\chi(g_{\bar\theta})$ for every χ.

Next, we write down the value of $\varphi_\xi^A(g_{\bar\theta})$, as proposed in the theorem for each series of representations.

To begin with, we consider $A=E$, and we note that, for $a\in E^\times$, we have that $a+\bar a=0$ if and only if $a=t\theta$ (resp. $a=t$), $t\in F^\times$, if char $F\neq 2$ (resp. char $F=2$), so that $\mathrm{Sol}_0^A(\infty,a)$ becomes the set of all z such that $N(z)=-N(t\theta)=t^2t_0$ (resp. $N(z)=N(t)=t^2$), i.e., $z=z_0tu$ (resp. $z=tu$) ($u\in U$), if char $F\neq 2$ (resp. char $F=2$). Hence, when char $F\neq 2$, for $\Lambda\in(E^\times)^\wedge$ we have that

$$\begin{aligned}\varphi_\Lambda^E(g_{\bar\theta}) &= \frac{-1}{2(q^2-1)}\sum_{t\in F^\times} S_\Phi^E(\infty,t\theta)\Lambda(t\theta)\\ &= \frac{-1}{2(q^2-1)}\sum_{t\in F^\times,u\in U}\overline{\Phi(z_0tu)}\Lambda(t)\Lambda(\theta) = -\frac{\Lambda(\theta)}{\Phi(z_0)}\delta_{\Phi,\Phi^q}\delta_{\Lambda_{|F^\times},\Phi_{|F^\times}}.\end{aligned}$$

When char $F=2$, we have that

$$\begin{aligned}\varphi_\Lambda^E(g_{\bar\theta}) &= \frac{-1}{2(q^2-1)}\sum_{t\in F^\times} S_\Phi^E(\infty,t)\Lambda(t)\\ &= \frac{-1}{2(q^2-1)}\sum_{t\in F^\times,\,u\in U}\overline{\Phi(tu)}\Lambda(t) = -\delta_{\Phi,\Phi^q}\,\delta_{\Lambda|F^\times,\Phi_{|F^\times}}.\end{aligned}$$

So, when Λ is the parameter of a discrete series representation in $\mathrm{Ind}_K^G(\Phi)$ we obtain that $\varphi_\Lambda^E(g_{\bar\theta})=\varphi_{\chi_\Lambda^{q-1}(g_{\bar\theta})}$

Now, we study the case $A=F\times F$. In this case, the computation is independent of the characteristic. We have that $a+\bar a=0$ if and only if $a=(b,-b)$, $b\in F$, and $\mathrm{Sol}^{F\times F}(\infty,(b,-b))=\{bu,\,u\in U\}$. Thus, $\varphi_{\alpha,\beta}^{F\times F}(g_{\bar\theta})=(1/(q^2-1))\sum_{b\in F^\times,\,u\in U}\overline{\Phi(b)\Phi(u)}\alpha(b)\beta(-b) = \delta_{\Phi,\Phi^q}\delta_{\Phi_{|F^\times},\alpha\beta}\beta(-1)$. So whenever (α, β) is the parameter of a principal series representation in $\mathrm{Ind}_K(\Phi)$, we obtain that $\varphi^{F\times F}_{\alpha,\beta}(g_{\bar\theta}) = \varphi_{\chi^{q+1}_{\alpha,\beta}}(g_{\bar\theta})$.

For the Steinberg representation, for char $F \neq 2$ we have that

$$\begin{aligned}\frac{1}{2}\Big[\sum_A \varphi^A_{\alpha_A}\Big] &= \frac{1}{2(q^2-1)}\Big[\sum_A \mathrm{sgn}(A) \sum_{a\in A^\times} S^A_\Phi(\infty, a)\alpha\circ n_A(a)\Big] \\ &= \frac{1}{2(q^2-1)}(-1)\Big[\sum_{t\in F^\times,\, u\in U} \overline{\Phi(z_0tu)}\alpha(t^2(-1)t_0) \\ &\qquad + \sum_{t\in F^\times,\, u\in U} \overline{\Phi(t)\Phi(u)}\alpha(-t^2)\Big] \\ &= \frac{1}{2}\delta_{\Phi,\,\Phi^q}\,\delta_{\Phi_{|F^\times},\alpha^2}\,\alpha(-1)\Big[1-\frac{\alpha(t_0)}{\Phi(z_0)}\Big].\end{aligned}$$

Therefore, whenever the representation π^q_α occurs in Ind(Φ) we obtain the desired equality. For the case char $F = 2$ we obtain

$$\begin{aligned}\frac{1}{2}\Big[\sum_A \varphi^A_{\alpha_A}\Big] &= \frac{1}{2(q^2-1)}\Big[(-1)\sum_{t\in F^\times, u\in U}\overline{\Phi(tu)}\alpha(t^2) + \sum_{t\in F^\times, u\in U}\overline{\Phi(t)\phi(u)}\alpha(t^2)\Big] \\ &= 0.\end{aligned}$$

Again, if π^q_α occurs in Ind(Φ) we obtain the inequality that we were looking for.

The computation for the case of a one-dimensional representation is analogous to the case of the Steinberg representation. This concludes the proof of Theorem 2 for $x = \bar\theta$.

For the next computation, we recall that the set of solutions to $x^2 + bx + c = 0$ $(b, c \in F)$ is $\{z, \bar z\}$ $(z \in E - F)$ if and only if char $F \neq 2$ and $b^2 - 4c \notin F^\sharp$, or if and only if char $F = 2$, $b \neq 0$, and $c/b^2 \notin F^\sharp$.

To complete the proof of Theorem 2, we fix $x \notin \{\theta, \bar\theta\}$. We begin to compute $\varphi^\Phi_\chi(g_x)$.

We first notice that $m_z g_x$ is never a scalar matrix. Thus, any matrix $m_z g_x$ is conjugated to one of the following:

$$\begin{pmatrix} a & a \\ 0 & a\end{pmatrix}(a\neq 0), \qquad \begin{pmatrix} a & 0 \\ 0 & d\end{pmatrix}(a \neq d), \qquad m_w(w \in E^\times - F^\times).$$

unipotent mod center case   hyperbolic case   elliptic case

Indeed, if char $F \neq 2$ the conjugacy class associated to $m_z g_x$ is determined according to whether $\delta := \delta(z, x) := (\mathrm{tra}(m_z g_x))^2 - 4\det(m_z g_x)$ is, respectively, 0, a nonzero square in F, of a nonsquare in F. Whereas, when char $F = 2$ the conjugacy class associated to $m_z g_x$ is determined according to $\mathrm{tra}(m_z g_x) = 0$ (unipotent mod center case) or $\mathrm{tra}(m_z g_x) \neq 0$ and $\det(m_z g_x)/(\mathrm{tra}(m_z g_x))^2$ is in $F^\sharp$ (hyperbolic case) or $\mathrm{tra}(m_z g_x) \neq 0$ and $\det(m_z g_x)/(\mathrm{tra}(m_z g_x))^2$ is not in $F^\sharp$ (elliptic case). Henceforth, for any two matrices A, B, the symbol $A \equiv B$ means that A is conjugated to B. We write "unipotent" instead of "unipotent mod center case," "hyperbolic" instead of "hyperbolic case," and "elliptic" instead of "elliptic case."

Because of (10) we have

$$\varphi_\chi(g_x) = \sum_{z:m_z g_x \equiv \mathrm{unip}} \cdots + \sum_{z:m_z g_x \equiv \mathrm{hyp}} \cdots + \sum_{z:m_z g_x \equiv \mathrm{ellip}} \cdots .$$

To determine when $m_z g_x$ is unipotent, hyperbolic, or elliptic, we set $x' = \frac{1}{2}(1 + x_2 - \theta(x_1/t_0)) = \frac{\theta - \bar{x}}{2\theta}$, if char $F \neq 2$, and $x' = \bar{x} + \theta$ for char $F = 2$.

Thus, we get, in any characteristic, $1 - \frac{1}{x'} = \frac{\bar{x}+\bar{\theta}}{\bar{x}+\theta}$. Since $\rho = N(\frac{x-\theta}{x-\bar{\theta}})$, it follows that

$$1 - \rho = \frac{\mathrm{Tr}(x') - 1}{N(x')} = \frac{x_2}{N(x')} = \frac{\det(g_x)}{N(x')},$$

in any characteristic. Hence, setting $z' = zx'$, for any $z \in E^\times$, we finally get

$$\det(m_z g_x) = (1-\rho)N(z'), \qquad \mathrm{tra}(m_z g_x) = \mathrm{Tr}(z'), \tag{10b}$$

in any characteristic.

By definition z, z' determine each other, and we have that $\overline{\Phi(z)} = \Phi(x')\overline{\Phi(z')}$. The above equations (10b) imply that $m_z g_x \equiv m_w$ if and only if

$$(1-\rho)N(z') = N(w) \quad \text{and} \quad \mathrm{Tr}(z') = \mathrm{Tr}(w), \tag{11}$$

that is, if and only if $z' \in \mathrm{Sol}^E(\rho, w)$. We notice that $w \neq \bar{w}$ because $m_z g_x$ is not a scalar matrix. Now, $\mathrm{Sol}^E(\rho, w)$ is nonempty if and only if $\delta := \mathrm{Tr}(w)^2 - 4\frac{N(w)}{1-\rho} \notin F^\sharp$ for char $F \neq 2$ (resp. $\delta := N(w)/(1-\rho)\,\mathrm{Tr}(w)^2 \notin F^\sharp$ for char $F = 2$).

Recall that $\mathcal{U}\colon E^\times \to E^\times$ is given by $\mathcal{U}(w) = w/\bar{w}$. Thus, $\mathrm{Tr}(w)^2/N(w) = \mathcal{U}(w) + \mathcal{U}(\bar{w}) + 2$. Hence, for char $F \neq 2$,

$$\delta = N(w)\left[\mathcal{U}(w) + \mathcal{U}(\bar{w}) - 2\frac{1+\rho}{1-\rho}\right] = N(w)\mathrm{Tr}\left[\mathcal{U}(w) - \frac{1+\rho}{1-\rho}\right].$$

Thus, $\mathrm{Sol}^E(\rho, w)$ is nonempty if and only if $N(w)\mathrm{Tr}[\mathcal{U}(w) - \frac{1+\rho}{1-\rho}] \notin F^\sharp$. Since $w \neq \bar{w}$, the number of solutions to (11) is either 0 or 2. Thus,

$$|\mathrm{Sol}^E(\rho, w)| = 1 - \epsilon\left(N(w)\mathrm{Tr}\left[\mathcal{U}(w) - \frac{1+\rho}{1-\rho}\right]\right).$$

Therefore, we have verified (2). Moreover, the contribution of the elliptic elements to the formula for $\varphi_\chi(g_x)$, when char $F \neq 2$, is

$$\frac{1}{(q^2-1)} \sum_{z:m_z g_x \equiv \text{elliptic}} \cdots = \frac{1}{(q^2-1)} \sum_{w \in E^\times,\, w \neq \bar{w}} \sum_{z:m_z g_x \equiv m_w} \overline{\Phi(z)} \chi(m_z g_x)$$
$$= \frac{1}{(q^2-1)} \Phi(x') \sum_{w \in E^\times, w \neq \bar{w}} \left[\sum_{z' \in Sol^E(\rho,w)} \overline{\Phi(z')} \right] \chi(m_w)$$
$$= \frac{1}{2(q^2-1)} \Phi(x') \sum_{w \in E^\times, w \neq \bar{w}} S_\Phi^E(\rho, w) \chi(m_w).$$

(Here we write $z' = x'z$ for any $z \in E^\times$, as above.)

For the case char $F = 2$, we have that $\mathrm{Sol}^E(\rho, w)$ is nonempty if and only if $N(w)/(1-\rho)\mathrm{Tr}(w)^2 \notin F^\sharp$. Thus, $|\mathrm{Sol}^E(\rho, w)| = 1 - \epsilon[((1-\rho)\mathrm{Tr}(\mathscr{U}(w)))^{-1}]$. Whence, we have (4), and when char $F = 2$, the contribution of the elliptic elements to the formula for $\varphi_\chi(g_x)$ is

$$\frac{1}{(q^2-1)} \sum_{z:m_z g_x \equiv \text{elliptic}} \cdots = \frac{1}{2(q^2-1)} \Phi(x') \sum_{w \in E^\times, w \neq \bar{w}} S_\Phi^E(\rho, w) \chi(m_w).$$

Next, we study the contribution of the unipotent matrices to the formula for $\varphi_\chi(g_x)$. The equation in m_z,

$$m_z g_x \equiv \begin{pmatrix} t & t \\ 0 & t \end{pmatrix},$$

is equivalent to

$$\mathrm{Tr}(z') = 2t, \quad N(z') = \frac{t^2}{1-\rho}. \tag{12}$$

Thus, $m_z g_x$ is conjugated to a unipotent matrix if and only if $\mathrm{Sol}^{F\times F}(\rho, (t,t))$ is nonempty. Now, whenever char $F \neq 2$, (12) has a solution z' if and only if $\delta(z') = \mathrm{Tr}(z')^2 - 4N(z')$ is not in $F^\sharp$. Since $\delta(z') = 4t^2(1 - \frac{1}{1-\rho}) = 4t^2(\frac{\rho}{\rho-1})$, we have that $|\mathrm{Sol}^{F\times F}(\rho, (t,t))| = 1 - \epsilon(\frac{\rho}{\rho-1})$. Therefore, we have (3), and we obtain, for char $F \neq 2$,

$$\frac{1}{q^2-1} \sum_{m_z g_x \equiv \text{unip.}} \cdots = \frac{1}{(q^2-1)} \Phi(x') \sum_{t \in F^\times} \left[\sum_{z' \in \mathrm{Sol}^{F\times F}(\rho,(t,t))} \overline{\Phi(z')} \right] \chi \begin{pmatrix} t & t \\ 0 & t \end{pmatrix}$$
$$= \frac{1}{q+1} \Phi(x')\, S_\Phi^{F\times F}(\rho, 1) \chi \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The last equality is due to $\mathrm{Sol}^{F\times F}(\rho, (t,t)) = t\, \mathrm{Sol}^{F\times F}(\rho, (1,1))$.

In the case char $F = 2$, to study the contribution of the unipotent elements to the formula, we need to find z' so that $\mathrm{Tr}(z') = 0$, $N(z') = t^2/(1-\rho)$. This forces $z' = \frac{t}{\sqrt{1-\rho}}$ and $|\mathrm{Sol}^{F\times F}(\rho,(t,t))| = 1$. Therefore, we have (5) and

$$\begin{aligned}\frac{1}{q^2-1}\sum_{m_z g_x \equiv \text{unip.}} \cdots &= \frac{1}{(q^2-1)}\Phi(x')\Big[\sum_{t\in F^\times}\overline{\Phi}\Big(\frac{t}{\sqrt{1-\rho}}\Big)\chi\begin{pmatrix} t & t\\ 0 & t\end{pmatrix}\Big]\\ &= \frac{1}{q^2-1}\Phi(x')\Big[\sum_{t\in F^\times} S_\Phi^{F\times F}(\rho,(t,t))\,\chi\begin{pmatrix} t & t\\ 0 & t\end{pmatrix}\Big]\\ &= \frac{1}{q+1}\Phi(x')\,\Phi\big(\sqrt{1-\rho}\big)\,\chi(Id)\,\chi\begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix}.\end{aligned}$$

Finally, the contribution for the hyperbolic elements $m_z g_x \equiv \begin{pmatrix} a & 0\\ 0 & d\end{pmatrix}$ gives rise to the equations

$$\mathrm{Tr}(z') = a + d, \quad N(z') = \frac{ad}{1-\rho}, \qquad a \neq d. \tag{13}$$

As before, when char $F \neq 2$, $\delta(z')$ must be a nonsquare and

$$\begin{aligned}\delta(z') &= \mathrm{Tr}(z')^2 - 4N(z') = (a+d)^2 - 4\frac{ad}{1-\rho}\\ &= ad\Big(\frac{(a+d)^2}{ad} - \frac{ad}{1-\rho}\Big) = ad\Big(\frac{a}{d} + \frac{d}{a} - 2\frac{(1+\rho)}{(1-\rho)}\Big).\end{aligned}$$

Therefore,

$$|\mathrm{Sol}^{F\times F}(\rho,(a,d))| = 1 - \epsilon\Big[N_{F\times F}(a,d)\Big(\mathrm{tra}_{F\times F}\Big(\mathcal{U}_{F\times F}(a,d) - \frac{1+\rho}{1-\rho}\Big)\Big)\Big].$$

Thus, we get

$$\begin{aligned}\frac{1}{q^2-1}\sum_{m_z g_x \equiv \text{hyper.}} \cdots &= \frac{1}{2(q^2-1)}\Phi(x')\\ &\quad\times \sum_{(a,d),a\neq d}\Big[\sum_{z'\in \mathrm{Sol}^{F\times F}(\rho,(a,d))}\overline{\Phi(z')}\Big]\chi\begin{pmatrix} a & 0\\ 0 & d\end{pmatrix}.\end{aligned}$$

Whence, for char $F \neq 2$, the contribution of the hyperbolic elements to $\varphi_\chi(g)$ is

$$\frac{1}{q^2-1}\sum_{m_z g_x \equiv \text{hyper.}} \cdots = \frac{1}{2(q^2-1)}\Phi(x')\sum_{(a,d),a\neq d,} S_\Phi^{F\times F}(\rho,(a,d))\chi\begin{pmatrix} a & 0\\ 0 & d\end{pmatrix}.$$

For the case char $F = 2$, the system (13) has a solution in z' if and only if $N(z')/\mathrm{Tr}(z')^2 \notin F^\sharp$. Now, $N(z')/\mathrm{Tr}(z')^2 = 1/(1-\rho)\mathrm{tra}_{F\times F}(\mathcal{U}(a,d))$. Therefore,

$$|\mathrm{Sol}^{F\times F}(\rho,(a,d))| = 1 - \epsilon[((1-\rho)\mathrm{tra}_{F\times F}(\mathcal{U}_{F\times F}(a,d)))^{-1}].$$

Thus, we get

$$\begin{aligned}\frac{1}{q^2-1}\sum_{m_z g_x \equiv \text{hyper.}} \cdots &= \frac{1}{2(q^2-1)}\Phi(x') \\ &\quad\times \sum_{(a,d), a\neq d}\left[\sum_{z'\in \mathrm{Sol}^{F\times F}(\rho,(a,d))} \overline{\Phi(z')}\right]\chi\begin{pmatrix} a & 0 \\ 0 & d\end{pmatrix} \\ &= \frac{1}{2(q^2-1)}\Phi(x')\sum_{(a,d), a\neq d} S_\Phi^{F\times F}(\rho,(a,d))\chi\begin{pmatrix} a & 0 \\ 0 & d\end{pmatrix}.\end{aligned}$$

Next, we spell out the formula $\varphi_\chi^\Phi(g_x)$ for each series of representations for $Gl(2,F)$, and we show that each of them is equal to $\varphi_\xi^A(g_x)$, as asserted in the theorem. To this end, we will use the table in [5, p. 64].

For the discrete series χ_Λ^{q-1} of parameter Λ, in any characteristic we have that

$$\varphi_{\chi_\Lambda^{q-1}}(g_x) = \sum_{m_z g_x \equiv elliptic} \cdots + \sum_{m_z g_x \equiv unipotent} \cdots.$$

Thus,

$$\begin{aligned}\varphi_{\chi_\Lambda^{q-1}}(g_x) &= \frac{1}{2(q^2-1)}\Phi(x')\sum_{w\in E^\times, w\neq \bar{w}} S_\Phi^E(\rho,w)(-1)(\Lambda+\Lambda^q)(w) \\ &\quad + \frac{1}{(q^2-1)}\Phi(x')\sum_{t\in F^\times} S_\Phi^{F\times F}(\rho,(t,t))(-\Lambda(t)).\end{aligned}$$

Since $S_\Phi^E(\rho,w) = S_\Phi^E(\rho,w^q)$ and $S_\Phi^E(\rho,t) = S_\Phi^{F\times F}(\rho,(t,t))$ for $t\in F^\times$, we have that

$$\varphi_{\chi_\Lambda^{q-1}}(g_x) = \frac{-1}{(q^2-1)}\Phi(x')\sum_{w\in E^\times} S_\Phi^E(\rho,w)\Lambda(w) = \varphi_\Lambda^E(g_x).$$

For a principal series representation $\chi_{\alpha,\beta}^{q+1}$, in any characteristic, we have that $\varphi_{\chi_{\alpha,\beta}^{q+1}}(g_x) = \sum_{m_z g_x \equiv \text{hyperbolic}} \cdots + \sum_{m_z g_x \equiv \text{unipotent}} \cdots$. Thus, we obtain

$$\begin{aligned}\varphi_{\chi_{\alpha,\beta}^{q+1}}(g_x) &= \frac{1}{2(q^2-1)}\Phi(x') \\ &\quad\times \sum_{(a,d), a\neq d} S_\Phi^{F\times F}(\rho,(a,d))[\alpha(a)\beta(d)+\alpha(d)\beta(a)] \\ &\quad + \frac{1}{(q^2-1)}\Phi(x')\sum_{t\in F^\times} S_\Phi^{F\times F}(\rho,(t,t))\alpha(t)\beta(t).\end{aligned}$$

Hence, we have that

$$\varphi_{\chi^{q+1}_{\alpha,\beta}}(g_x) = \frac{1}{(q^2-1)}\Phi(x')\sum_{(a,d)} S_{\Phi}^{F\times F}(\rho,(a,d))\alpha(a)\beta(d) = \varphi^{F\times F}_{\alpha,\beta}(g_x).$$

For the Steinberg representation χ^q_α, in any characteristic, we have that $\varphi_{\chi^q_\alpha}(g_x) = \sum_{\text{ellip.}}\cdots + \sum_{\text{hyper.}}\cdots$. Thus, we obtain

$$\varphi_{\chi^q_\alpha}(g_x) = \frac{1}{2(q^2-1)}\Phi(x')\sum_{w\in E^\times, w\neq\bar{w}} S^E_\Phi(\rho,w)(-1)\alpha(N(w))$$
$$+\frac{1}{2(q^2-1)}\Phi(x')\sum_{(a,d),a\neq d} S^{F\times F}_\Phi(\rho,(a,d))\alpha(ad).$$

Thus, $\varphi_{\chi^q_\alpha}(g_x)$ is given as in the formula proposed in Theorem 2.

For the case of the spherical functions associated to the one-dimensional representations, the computation is the same as in the case of the Steinberg representations. ∎

Remark 2. Theorem 2 generalizes results of [1, 7, 11] and unpublished results of both authors for class one representations.

8. SOME CHARACTER IDENTITIES

In this section we deduce some identities involving sign characters from the fact that certain irreducible representations π do not occur in $\mathrm{Ind}_K^G(\Phi)$, so that $\varphi^\Phi_\pi(g) = \sum_{h\in K}\overline{\Phi(h)}\chi_\pi(hg) \equiv 0$. From now on we assume that the characteristic of the field F is not equal to 2. Let A be a two-dimensional associative semisimple algebra over F. As in Section 6, tr_A, sgn_A, n_A denote the trace, sign, and norm of A over F. Let $U(A) := \{x \in A : n_A(x) = 1\}$. Notice that $|U(A)| = q - \mathrm{sgn}_A$. Let $\epsilon_{U(A)}$ be the character of $U(A)$ obtained by lifting the nontrivial character for $U(A)/U(A)^2$. Let ϵ_F be the order 2 character for $F^\times$ extended by zero to F. We notice that, for $A = F\times F$, $\epsilon_{U(A)}(-1)\epsilon_F(-1) = 1$, whereas if $A = E$, $\epsilon_{U(A)}(-1)\epsilon_F(-1) = -1$. That is, $\epsilon_{U(A)}(-1)\epsilon_F(-1) = \mathrm{sgn}_A$. Let

$$\varpi_A(s) := \sum_{u\in U(A)} \epsilon_{U(A)}(u)\epsilon_F(\mathrm{tr}_A(u-s)), \qquad s\in F.$$

Then, we have

THEOREM 3. (a) $\varpi_A(-s) = \epsilon_{U(A)}(-1)\epsilon_F(-1)\varpi_A(s) = \mathrm{sgn}_A\varpi_A(s)$.

(b) $\varpi_A(s) = -(1+\mathrm{sgn}_A)$, if $s\neq\pm1$.

(c) $\varpi_A(1) = q-(1+\mathrm{sgn}_A) = |U(A)|-1$.

The proof of (a) is obvious. We now show (b) and (c). Let F, E, U, $\mathcal{U}$, N, Tr, ϵ have the same meaning as in the previous paragraphs. Thus, $\epsilon = \epsilon_F$. Also, $U(F \times F) = \{(t, \frac{1}{t}) : t \in F^\times\}$ and $\epsilon_{U(F\times F)}(t, \frac{1}{t})\} = \epsilon(t)$. Let $\epsilon_0 := \epsilon_{U(E)}$. Written explicitly, Theorem 3 is equivalent to the following four statements for $s \in F$:

$$\sum_{t\in F^\times} \epsilon(t)\epsilon(t + \frac{1}{t} - 2s) = -2, \qquad \text{for } s \neq \pm 1 \tag{i.a}$$

$$\sum_{t\in F^\times} \epsilon(t)\epsilon(t + \frac{1}{t} - 2s) = q - 2, \qquad \text{for } s = \pm 1 \tag{i.b}$$

$$\sum_{u\in U} \epsilon_0(u)\epsilon(\mathrm{Tr}(u) - 2s) = 0, \qquad \text{for } s \neq \pm 1 \tag{ii.a}$$

$$\sum_{u\in U} \epsilon_0(u)\epsilon(\mathrm{Tr}(u) - 2s) = \pm q, \qquad \text{for } s = \pm 1. \tag{ii.b}$$

We first show (i.a), (ii.a). Let $x \in \mathcal{H}$ and set as before $D(x, \theta) = \rho$ and $s = \frac{1+\rho}{1-\rho}$. To begin with, we assume $\rho \notin \{0, \infty\}$. Hence $x \neq \pm\theta$, $s \neq \pm 1$. We now prove

$$\sum_{u\in U} \epsilon_0(u)\epsilon\left(\mathrm{Tr}\left[u - \frac{1+\rho}{1-\rho}\right]\right) - \sum_{t\in F^\times} \epsilon(t)\epsilon\left(t + \frac{1}{t} - 2\frac{1+\rho}{1-\rho}\right) = 2. \tag{14}$$

From Table I it follows that π_1^q is not a subrepresentation of $L^2(\mathcal{H})$. Hence, we have that $0 \equiv \sum_{h\in K} \chi_{\pi_1^q}(hg_x) = \varphi^1_{\pi_1^q}(g_x)$. On the other hand, we know that

$$\varphi^1_{\pi_1^q}(g_x) = \frac{1}{2(q^2-1)}\left[-(q^2-q) + \sum_{w\neq\bar{w}} \epsilon\left(N(w)\mathrm{Tr}\left[\mathcal{U}(w) - \frac{1+\rho}{1-\rho}\right]\right)\right]$$
$$+\frac{1}{4(q^2-1)} \sum_{a\neq d, a, d\in F^\times} 2(\mathbf{1} - \epsilon)\left(ad\left(\frac{a}{d} + \frac{d}{a}\right) - 2\frac{1+\rho}{1-\rho}\right).$$

That is,

$$\varphi^1_{\pi_1^q}(g_x) = \frac{1}{2(q^2-1)}\left[-(q^2-q) + (q-1)\sum_{u\in U, u\neq 1} \epsilon_0(u)\epsilon\left(\mathrm{Tr}\left[u - \frac{1+\rho}{1-\rho}\right]\right)\right]$$
$$+\frac{1}{2(q+1)}\left[(q-2) - \sum_{t\neq 1} \epsilon(t)\epsilon\left(t + \frac{1}{t} - 2\frac{1+\rho}{1-\rho}\right)\right].$$

To deduce the second equality we used that $E^\times - F^\times = \cup_{\{u\in U,\, u\neq 1\}} e_u F^\times$, where we choose $e_u \in E^\times$ such that $\mathcal{U}(e_u) = u$, and on the second sumand we performed the change of variable $a = a$, $d = at$. Thus, we obtain

$$0 = -2 + \sum_{u\in U, u\neq 1} \epsilon_0(u)\epsilon\left(\mathrm{Tr}\left[u - \frac{1+\rho}{1-\rho}\right]\right) - \sum_{t\neq 1} \epsilon(t)\epsilon\left(t + \frac{1}{t} - 2\frac{1+\rho}{1-\rho}\right),$$

so we have proved (14). Hence, we have, for $s \neq \pm 1$, that

$$\sum_{u\in U} \epsilon_0(u)\epsilon(\mathrm{Tr}(u) - 2s) - \sum_{t\in F^\times} \epsilon(t)\epsilon\left(t + \frac{1}{t} - 2s\right) = 2. \tag{15}$$

Replacing, in each summand in (15), s by $-s$, u by $-u$, and t by $-t$, and recalling that $\epsilon_0(-1) = -\epsilon(-1)$, we obtain, for $s \neq \pm 1$, the equality

$$-\sum_{u\in U} \epsilon_0(u)\epsilon(\mathrm{Tr}(u) - 2s) - \sum_{t\in F^\times} \epsilon(t)\epsilon\left(t + \frac{1}{t} - 2s\right) = 2.$$

Thus, we have proved (i.a), (ii.a) for $s \neq \pm 1$. To show (i.b), (ii.b) we notice that the function in (i.a) is even and that the function in (ii.a) is odd. Hence, we are left to consider $s = -1$. Now, for $t \in F^\times$, $\epsilon(t)\epsilon(t + \frac{1}{t} + 2) = \epsilon((t+1)^2) = 1$ unless $t = -1$. Therefore, $\sum_{t\in F^\times} \epsilon(t)\epsilon(t + \frac{1}{t} + 2) = q - 2$, and hence (i.b) follows. To conclude the verification of (ii.b) we notice that for $u \in U$, we have $\mathrm{Tr}(u) + 2 = N(u+1)$ and that $\epsilon(N(y)) = \epsilon_E(y)$, for $y \in E^\times$. Here ϵ_E is the function on $E^\times$, which is equal to 1 on $(E^\times)^2$ and -1 elsewhere. For $u \in U$ we write $u = \frac{z}{\bar{z}}$ for a suitable $z \in E^\times$. Thus, $z + \bar{z}$ is a square in E and

$$\begin{aligned} \epsilon(\mathrm{Tr}(u) + 2) &= \epsilon(N(u+1)) = \epsilon_E(u+1) = \epsilon_E\left(\frac{z}{\bar{z}} + 1\right) \\ &= \epsilon_E(\bar{z}^{-1})\epsilon_E(z + \bar{z}) = \epsilon_E(z) = \epsilon_0(\mathcal{U}(z)) = \epsilon_0(u), \end{aligned}$$

unless $u = -1$, in which case $\epsilon(\mathrm{Tr}(u) + 2) = 0$. It follows that

$$\sum_{u\in U} \epsilon_0(u)\epsilon(\mathrm{Tr}(u) + 2) = \sum_{u\in U - \{-1\}} \epsilon_0(u)\epsilon_0(u) = q,$$

as claimed. ∎

As a consequence we have

COROLLARY 2.

$$\sum_{u\in U} \epsilon(\mathrm{Tr}(u) - 2s) = -\sum_{t\in F^\times} \epsilon\left(t + \frac{1}{t} - 2s\right), \qquad \textit{if } s \neq \pm 1 \tag{iii}$$

$$\sum_{u\in U} \epsilon(\mathrm{Tr}(u) \pm 2) = -\sum_{t\in F^\times} \epsilon\left(t + \frac{1}{t} - \pm 2\right) = -\epsilon(-1). \tag{iv}$$

To show (iii) we notice that Table I implies that π_ϵ^1 does not occur in $L^2(\mathscr{H})$. Hence, in the same way as in the previous paragraph, we obtain the desired equality. However, a direct proof follows readily from $\sum_{t\in F^\times}\epsilon(t) = 0$, $F = \{\mathrm{Tr}(u) - s, u \in U\} \cup \{t + \frac{1}{t} - s,\ t \in F^\times\}$, $\{\mathrm{Tr}(u) - s, u \in U\} \cap \{t + \frac{1}{t} - s, t \in F^\times\} = \{2 - s, -2 - s\}$, and the fact that each element of F not in the intersection is obtained twice. Thus, we have finished the proof of (iii). To verify (iv), we notice $\sum_{t\in F^\times}\epsilon(t + \frac{1}{t} + 2) = \sum_{t\in F^\times}\epsilon(t)\epsilon((t+1)^2) = \sum_{t\neq -1}\epsilon(t) = -\epsilon(-1)$. ∎

Remark 3. If we try to exploit the fact that neither $\pi^1_{\epsilon\alpha}$ nor π^q_α occurs in $\mathrm{Ind}_K^G(\alpha\circ N)$, we obtain no new identities. If we compute the class 1 spherical function for the trivial representation, we obtain another proof of (i), (ii).

Remark 4. From Section 7 we have that $\varphi^1_{\chi^{q+1}_{\epsilon,\epsilon}}(g_x) = \frac{1}{q+1}\sum_{t\in F^\times}\epsilon(t + \frac{1}{t} + -2\frac{1+\rho}{1-\rho})$. Also, if $\omega \in \hat{U}$ is such that $\omega \neq \omega^{-1}$, then $\Phi = \omega \circ \mathscr{U}$ satisfies $\Phi \neq \Phi^q$, and the class 1 spherical function attached to π_Φ^{q-1} is $\varphi^1_{\chi_\Phi^{q-1}} = \frac{1}{q+1}\sum_{u\in U}\epsilon(\mathrm{Tr}(u - \frac{1+\rho}{1-\rho}))\epsilon_0(u)\omega(u)$. We could imagine that $Gl(2, F)$ has two class 1 limits of discrete series representations, $\pi^{q-1}_{1\circ\mathscr{U}}$, $\pi^{q-1}_{\epsilon_0\circ\mathscr{U}}$. If so, (iii) express that $\varphi^1_{\chi^{q-1}_{\epsilon_0\circ\mathscr{U}}}(g_x) = -\varphi^1_{\chi^{q+1}_{\epsilon,\epsilon}}(g_x)$, for $x \neq \pm\theta$, and (ii) says that $\varphi^1_{\chi^{q-1}_{1\circ\mathscr{U}}}(g_x) = 0$, for $x \neq \pm\theta$. Finally, (i) reads $\varphi^1_{\chi^{q+1}_{1,1}}(g_x) = \frac{-2}{q+1}$, for $x \neq \pm\theta$.

Remark 5. For $\Phi \neq \Phi^q$, it follows from Table 1 that the discrete series representation π_Φ^{q-1} does not occur in $\mathrm{Ind}_K^G(\Phi)$. Therefore, for $\rho \notin \{0, \infty\}$, from Section 7 we obtain

$$\sum_{w\in E^\times} S_\Phi^E(\rho, w)\Phi(w) = 0,$$

whereas, for $\Phi = \Phi^q$, i.e., $\Phi = \phi_0 \circ N$, with $\phi_0 \in (F^\times)^\wedge$, we have

$$\frac{1}{(q^2-1)}\sum_{w\in E^\times} S_\Phi^E(\rho, w)\Phi(w) = \phi_0(x_2).$$

For this we recall that $\phi_0(N(\frac{x+\theta}{2\theta})) = \phi_0(x_2/(1-\rho))$. Thus,

$$\frac{1}{(q^2-1)}\sum_{w\in E^\times} S_\Phi^E(\rho, w)\Phi(w)$$
$$= \frac{1}{(q^2-1)}\phi_0\left(\frac{x_2}{1-\rho}\right)\sum_{w\in E^\times}(\overline{\Phi(z'_w)} + \overline{\Phi(\bar{z'_w})})\phi_0(N(w)) = \natural.$$

Here, z'_w is a solution to $N(z'_w) = \frac{N(w)}{1-\rho}$ and $\mathrm{Tr}(z'_w) = \mathrm{Tr}(w)$. Thus, $\Phi(z'_w) = \phi_0(N(z'_w)) = \phi_0(N(w))\overline{\phi_0(1-\rho)}$.

Now, since $s = \frac{1+\rho}{1-\rho}$, we have that

$$\begin{aligned} \natural &= \frac{1}{(q^2-1)} \phi_0(x_2) \sum_{w \in E^\times} 1 - \epsilon(N(w))\, \epsilon(\mathrm{Tr}(\mathcal{U}(w) - s)) \\ &= \frac{1}{(q^2-1)} \phi_0(x_2) \big[q^2 - 1 - \sum_{w \in E^\times} \epsilon(N(w))\, \epsilon(\mathrm{Tr}(\mathcal{U}(w) - s)) \big] \\ &= \frac{1}{(q^2-1)} \phi_0(x_2) \big[q^2 - 1 - \sum_{t \in F^\times, u \in U} \epsilon(N(e_u t))\, \epsilon(\mathrm{Tr}(\mathcal{U}(e_u t) - s)) \big] \\ &= \frac{1}{(q^2-1)} \phi_0(x_2) \big[(q^2 - 1) - (q-1) \sum_{u \in U} \epsilon_0(u)\, \epsilon(\mathrm{Tr}(u - s)) \big] = \phi_0(x_2). \end{aligned}$$

Remark 6. Let b be a nondegenerate bilinear form on a finite-dimensional vector space V over the field $\mathbb{F}_{p^m}$. Let G be the isometry group of the "$\mathbb{F}_{p^m}$-valued metric" $d(x, y) = b(x - y, x - y)$. We fix $r \in \mathbb{F}_{p^m}, r \neq 0, 1$. We recall for $f \in L^2(V)$ the mean value linear operator $M_r(f)(x) = \sum_{y \in V, d(x,y)=r} f(y)$. It is clear that M_r lies in the commuting algebra for the natural representation of G in $L^2(V)$. In [9] the authors have proved the following: If $[\mathbb{F}_{p^m} : \mathbb{Z}/p\mathbb{Z}] > 1$ or $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ and V is odd dimensional, then the algebra spanned by M_r is a proper subalgebra of the whole algebra of intertwining operators. This also holds if $p = 3$. Whereas for $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ and V even dimensional, the whole intertwining algebra is spanned by M_r. Our proof is based upon a study of spherical functions on G. We believe that a similar result holds for the homographic action of $Gl(2, \mathbb{F}_{p^m})$ on $L^2(\mathcal{H})$.

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